Logic teaching in the 21st century

La enseñanza de la lógica en el siglo 21

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Resumen: En la actualidad, estamos mucho mejor preparados para dejar que los hechos se revelen ellos mismos a nosotros, en vez de ignorarlos o intentar obstinadamente hacerlos entrar en moldes preconcebidos. Ya dejamos de sentirnos incómodos ante nuestros estudiantes insistiendo, por ejemplo, que "algunos X son Y" significa lo mismo que "algún X es Y" y añadiendo, sin gran convicción, la frase "para los fines de la lógica" cuando encontramos resistencia.

La enseñanza de la lógica en este siglo puede explotar el nuevo espíritu de objetividad, humildad, claridad, observacionalismo, contextualismo, tolerancia y pluralismo. En consecuencia, la enseñanza de la lógica en este siglo puede acelerar el declive o, al menos, frenar la expansión del espíritu recurrente de subjetividad, intolerancia, ofuscamiento y relativismo.

Además de este nuevo espíritu, ha habido en la lógica, en su historia y en la filosofía avances significativos que podrían mejorar radicalmente la enseñanza de la lógica. Un ejemplo más bien notable es el hecho de que el proceso de refinamiento de la terminología lógica ha sido productivo. Los futuros estudiantes de lógica dejarán ya de ser agobiados por el lastre de una terminología oscura y podrán ser capaces de leer, pensar, hablar y escribir sobre lógica de una manera más cuidadosa y gratificante.

Estrechamente relacionado con esto último se encuentra el uso y el estudio crecientes del lenguaje natural enriquecido con variables como, por ejemplo: "Toda proposición x que implica alguna proposición y que es falsa implica también alguna proposición z que es verdadadera".

Otro desarrollo positivo es la culminación del lento rechazo del logicismo. El profesor ya no se siente impedido de hacer uso de ejemplos de la aritmética y el álgebra, temiendo estar adoctrinando a los estudiantes con la idea de que toda verdad matemática es una tautología y que toda falsedad matemática es una contradicción.

Un desarrollo positivo más es la separación de las leyes de la lógica y de las llamadas verdades lógicas, esto es las tautologías. Actualmente podemos enseñar la independencia lógica de las leyes de tercero excluso y no contradicción sin temer que los estudiantes hayan sido adoctrinados con la idea de que toda ley de la lógica es una tautología y que toda falsedad lógica es una contradicción. Esta separación permite al profesor de lógica aplicar la lógica en la clarificación de las leyes de la lógica.

Este texto explora los puntos mencionados, los cuales aplican por igual en cursos de primero, segundo y tercer nivel, esto es "pensamiento crítico", "lógica deductiva" y "lógica simbólica".

Palabras clave: enseñanza de la lógica, pluralismo, terminología lógica, proposiciones matemáticas, proposiciones lógicas.

Abstract: Today we are much better equipped to let the facts reveal themselves to us instead of blinding ourselves to them or stubbornly trying to force them into preconceived molds. We no longer embarrass ourselves in front of our students, for example, by insisting that "Some Xs are Y" means the same as "Some X is Y", and lamely adding "for purposes of logic" whenever there is pushback.

Logic teaching in this century can exploit the new spirit of objectivity, humility, clarity, observationalism, contextualism, tolerance, and pluralism. Accordingly, logic teaching in this century can hasten the decline or at least slow the growth of the recurring spirit of subjectivity, intolerance, obfuscation, and relativism.

Besides the new spirit there have been quiet developments in logic and its history and philosophy that could radically improve logic teaching. One rather conspicuous example is that the process of refining logical terminology has been productive. Future logic students will no longer be burdened by obscure terminology and they will be able to read, think, talk, and write about logic in a more careful and more rewarding manner.

Closely related is increased use and study of variable-enhanced natural language as in "Every proposition x that implies some proposition y that is false also implies some proposition z that is true".

Another welcome development is the culmination of the slow demise of logicism. No longer is the teacher blocked from using examples from arithmetic and algebra fearing that the students had been indoctrinated into thinking that every mathematical truth was a tautology and that every mathematical falsehood was a contradiction.

A further welcome development is the separation of laws of logic from so-called logical truths, i.e., tautologies. Now we can teach the logical independence of the laws of excluded middle and non-contradiction without fear that students had been indoctrinated into thinking that every logical law was a tautology and that every falsehood of logic was a contradiction. This separation permits the logic teacher to apply logic in the clarification of laws of logic.

This lecture expands the above points, which apply equally well in first, second, and third courses, i.e. in "critical thinking", "deductive logic", and "symbolic logic".

Keywords: logic teaching, pluralism, logical terminology, mathematical propositions, logical proposition.

Logical thinking in mathematics can be learned only by observation and experience. In fact, the ability to reason correctly and to understand correct reasoning is itself a prerequisite to the study of formal logic.

—SOLOMON FEFERMAN,
The Number Systems. 1964.

Introduction

The plan of this lecture is to expand each of the six themes contained in the abstract, each with its own section. Each such thematic section begins with a quote from the abstract. Within each of the thematic sections, connections will be made to the other sections and to the references. None of the sections are definitive: all raise more issues than they settle. This is in keeping with the new spirit treated in the next section below. Logic teachers in the 21st century no longer have to pretend that logic is a completed monolith or seamless tapestry of established

truths—or even that it is moving toward being such. New knowledge reveals new awareness of old ignorance. New knowledge also begs many questions. Can this result be improved? How can this result be applied? And many more. The goals of logic study are not limited to acquisition of truths but include acquisition of expertise ("Investigating knowledge and opinion", Corcoran-Hamid, 2015).

Moreover, logic teachers do not need to pretend to be inculcating truths or even to be telling the truth to their students. My 1999 essay "Critical thinking and pedagogical license", written to be read by students of logic, makes it clear that there is room in logic teaching for telling untruths and for letting the students in on the fact that effective teaching requires deviation from fact.

Like other sciences, there are five distinct kinds of knowledge in logic to be shared with students— not imparted to them: objectual, operational, propositional, hypothetical, and expert. Briefly,

objectual knowledge is *of* objects in the broad sense including individuals, concepts, processes, etc. Operational knowledge, or know-how, includes ability to observe, judge, deduce, etc. Propositional knowledge, or know-that, is knowing a proposition to be true or to be false. The expression *hypothetical knowledge* may be new to some. In the sense used here, I define *hypothetical* knowledge as knowledge of the "openness" of unsettled propositions and unsolved problems. Paradoxically put, hypothetical knowledge is knowledge of what is not knowledge, knowing where the uncharted territory is: for example, knowing of words whose meanings are not clear, knowing of propositions not known to be true or false, knowing of arguments not known to be valid or invalid, the list goes on and on.

This definition connects with using the noun *hypothesis* for "proposition not known to be true and not known to be false": we have no other word for this important concept. Although every proposition is either true or false, not every proposition is either known to be true or known to be false. Using this terminology, every proposition is either known to be true, known to be false, or a hypothesis.

Experts are valued for sharing their "ignorance"—which is a paradoxical way of saying that they are valued for revealing what they *don't know*—their hypothetical knowledge. In fact, experts are often valued as much for revealing what they *don't know* as for revealing what they *do know*—their propositional knowledge (Corcoran-Hamid, 2015).

Expertise, the fifth kind of knowledge, includes the practical and theoretical experience acquired over years of engagement with a discipline's reality. It includes the expert's feel for the subject and the expert's engagement with the reality the subject is about. Moreover it unifies and inter-relates the other four kinds of knowledge. The expert's hypothetical knowledge is one of the fuels that keep a discipline alive and growing.

The recognition of the variety of kinds of knowledge alerts students of what they have and what they are gaining; it also alerts them of what they might be missing and what their textbooks might be missing. In earlier times, only two of these five were explicitly recognized and even then not to the extent recognized today. For example, Galen recognized only a kind of objectual knowledge—of "universals" such as "human", "dog", and "olive"—and propositional knowledge—such as that the sun is hot (Galen 200? /1964: 31f, 55f). Over a millennium later, William of Ockham was in the same rut (William of Ockham 1330?/1990: 18ff). Overtones of this epistemic dichotomy can be found today, for example, in the dichotomy of "primitive notions" and "primitive sentences" in Tarski 1941/1995 and in Tarski repeated juxtaposition of definability with provability. Today we recognize much more. For example, the capacity to generate sentences is a kind of operational knowledge and the knowledge of "primitive rules" is in a way objectual and in a way operational.

§1. Objectivity and pluralism

Of that which receives precise formulation in mathematical logic, an important part is already vaguely present as a basic ingredient of daily discourse. The passage from non-mathematical, non-philosophical common sense to the first technicalities of mathematical logic is thus but a step, quickly taken. Once within the field, moreover, one need not travel to its farther end to reach a frontier; the field is itself a frontier, and investigators are active over much of its length. Even within an introductory exposition there is room for novelties which may not be devoid of interest to the specialis1.

—QUINE 1940, *Preface.*

Today we are much better equipped to let the facts reveal themselves to us instead of blinding ourselves to them or stubbornly trying to force them into preconceived molds. We no longer embarrass ourselves in front of our students, for example, by insisting that 'Some Xs are Y' means the same as 'Some X is Y', and lamely adding "for purposes of logic" when there is pushback.

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humility, clarity, observationalism, contextualism, tolerance, and pluralism. Accordingly, logic teaching in this century can hasten the decline or at least slow the growth of the recurring spirit of subjectivity, intolerance, obfuscation, and relativism.

Wishful thinking, a close friend of laziness and a sworn enemy of objectivity, has played such an embarrassing role in the history of logic that many of us cringe at the mere hint of its appearance. The transition from the feeling "it would be nice if all Xs were Ys" to the belief "certainly all Xs are Ys" is so easy it sometimes feels like an implication. And when it becomes too obvious that not all Xs are Ys, then is the time to drag out "certainly all Xs are reducible to Ys" or "certainly all Xs are transformable into Ys" or "Xs may be regarded as Ys". Rarely is the point made that when we are told explicitly that Xs may be regarded as Ys, there is at least the suggestion—if not a tacit admission—that Xs are not Ys.

By the way, to see that 'Some Xs are Y' does not mean the same as 'Some X is Y', one may notice that "Some prime numbers are even" is false: 2 is the only prime number that is even: no two prime numbers are even. But, "Some prime number is even" is true: the proexample is 2. (See Corcoran 2005: "Counterexamples and Proexamples"). To be explicit, "Some prime numbers are even" amounts to "Two or more prime numbers are even". Although in general 'some' with a singular amounts to 'at least one', nevertheless with a plural it amounts to 'two or more': the meaning of 'some' is context dependent. There are many other examples. From set theory we have the truth "Some set is empty" juxtaposed with the falsehood "Some sets are empty". Many otherwise excellent texts overlook this point. (See Cohen-Nagel 1934/1993: 42ff.*)

We no longer regard, for example, 'Every X is Y"—where 'Y' must stand for an adjective and 'is' indicates predication—as interchangeable with 'Every X is a Y"—where 'Y' must stand for a noun and 'is' indicates identity. Russell, Parry, Smiley, and others all arrived at the same conclusion. (See Corcoran 2008a: "Aristotle's many-sorted logic")

And we no longer pretend that the two-word expression 'is a' before a *common* noun expresses the

membership relation and that the *common* noun following is really a *proper* name of a class. See my 2013 "Errors in Tarski's 1983 truth-definition paper". The *is* of identity can make a predicate out of a *proper name* as in 'two plus one is three', where 'two plus one' is the subject and 'is three' the predicate. The *is* of predication can make a predicate out of an adjective as in 'two plus one is odd' where 'two plus one' is the subject and 'is odd' the predicate. A verb phrase for membership—such as 'belongs to' or 'is a member of' resembles the *is* of identity in that it can make a predicate out of a proper name as in 'two plus one belongs to the class of odd numbers' where 'two plus one' is the subject and 'belongs to the class of odd numbers' the predicate. But the proper name must be a name of a class.

We no longer try to "reduce" one of these three to one of the other two. The question of what if anything one of these has in common with either of the other two we leave to future logicians; the teacher need not pretend to know the last word.

We no longer call the adverb 'not' a conjunction or a connective, and we don't force it to mean "non" or "it is not the case that" or, even worse, "it is false that". Of course there are cases where 'not' is naturally interchangeable with 'it is not that' and 'it is not the case that': in front of 'every' as in 'not every prime number is odd'. But such situations are rare: 'not some integer is divisible by zero' is ungrammatical but 'no integer is divisible by zero' is true, of course.

We no longer say that the word 'nothing' is a name of the null set, or worse, a name of the number zero. We no longer use 'equals' to mean "is": (2 + 3) is 5; there is only one integer between 4 and 6—call it '(2 + 3)', call it '5', or call it by one of its other names. (See Corcoran-Ramnauth 2013: "Equality and identity"). Using 'equals' for 'is' in arithmetic may be a vestige of a time when people thought that (2 + 3) wasn't 5 itself, but only an equal of 5. And that mistake may have been reinforced by failing to make the use-mention distinction: the seven-character name '(2 + 3)' isn't the one-character name '5', but they name the same number—which some people might regard as a kind of equality. Tarski

discusses these mistakes in 1941/1995. When '=' is used for identity as opposed to equality, it would be better to call it *the is sign* and not *the equals sign.* We try to avoid expressions that encourage or even tolerate distorted views of the logical structure of language and we encourage our students to point out such expressions.

The string '(2 + 3)' has seven-characters: two parentheses, two digits, one plus sign, and two spaces. See the 1974 "String Theory" and the 2006 "Schemata".

If the logic you know exhausts all logic, your work learning logic is finished. And if you believe that the logic you know exhausts all logic, why should you look for places it doesn't work? After all, you are sure there are none. And when doubts creep in, apologetics and rationalization come to the rescue; and if you are desperate, invoking famous authorities might help.

My primary goal in logic teaching is to connect the students to the reality logic is about, not to indoctrinate the students in the opinions of famous logicians or to drill them in the currently fashionable manipulations. The aim is to bring out the student's native ability to make autonomous judgments and perhaps correct or even overthrow the current paradigms—not to swell the ranks of orthodoxy. Even worse than the enthusiastic orthodox logicians are those who lack a sense of logical reality and who therefore treat logic like fiction, spinning out one new artificial system after the other, all equally empty.

What do I mean by logical reality? What do I mean by physical reality? What do I mean by mathematical reality? What do I mean by reality? A "formal definition" is out of the question, but helpful things can be said. In keeping with normal usage, reality is what a person refers to in making an objective judgment. There are as many aspects to logical reality as there are categories of logical judgments. (See Corcoran 2009: "Sentence, proposition, judgment, statement, fact"). I asked Frango Nabrasa how he explains reality to people uncomfortable about the word 'reality'. His answer: "Reality is what people agree about when they actually agree and what people disagree about when they actually disagree". For uses of the word 'reality' in a logical context see, e.g., Russell's *Introduction to* (Wittgenstein 1922).

How is the reality that logic studies accessed? The short answer is "through its applications". A longer answer can be inferred from my 1973 article "Gaps between logical theory and mathematical practice".

The applications of logic are to living sciences, technologies, humanities, and disciplines—a point emphasized by Tarski, Henkin, and others in the Berkeley Logic and Methodology Group. Before any logic is discussed in the classroom some content should be presented, preferably content already familiar to the student or, if not familiar, useful and easily grasped. I have in mind arithmetic, algebra (or analysis), geometry, set theory, class theory, string theory (syntactics), zoology, botany, and—perhaps paradoxically—logic itself.

In particular, before a symbolic argument schema is presented, a discipline or disciplines and concrete arguments instantiating that schema should be presented. Of the various ways of presenting an argument perhaps the one least open to misinterpretation is the *premises-line-conclusion format* which consists in listing the premises followed by a line followed by the conclusion. There is no justification, other than mindless adherence to tradition, for using an inferential adverb such as 'therefore', 'hence', 'so', or the triple-dot therefore sign '∴' to mark the conclusion in a presentation of an argument. This confuses the mere presentation of an argument for consideration with the statement of its validity. We need to present arguments without seeming to claim their validity. Actually, use of an inferential adverb is even worse than that: besides claiming that the conclusion follows it seems to claim the truth of the premises. See my 1973 "Meanings of implication", which has been translated into Spanish.

Here is what I mean: concrete arguments from arithmetic, geometry, set theory, and logic are presented first and then some related schemata are given. (See Corcoran 2006: "Schemata"). Incidentally, in

this paper it looked nicer to underline the last premise before the conclusion than to make a line after the last premise. This will not work for zero-premise arguments. Another device that is handy is to prefix the conclusion with a special character having inferential connotations, for example, a question mark.

Every number divides itself.
Every even number divides itself.

Every triangle resembles itself.
Every equilateral triangle resembles itself.

Every set contains itself.
Every finite set contains itself.

Every proposition implies itself.
Every false proposition implies itself.

For future reference below, note that the above four arguments are in the same form. It will be important to remind ourselves of one of the ways an argument can be used as a template for generating the others. This method will be form-preserving: it generates from one argument new arguments having the same form. The simplest form-preserving transformation is the operation of substituting one new non-logical term for every occurrence of a given non-logical term. By 'new' here is meant "not already occurring in the argument operated on" and, of course, the semantic category of the new term must be the same as the one it replaces. For example, "number" can replace "integer" but it cannot replace "one", "even", "divides", "square-root", "plus", etc.

The operation just described is called *one-new-term-substitution.* Every argument obtained from a given argument by a finite sequence of one-new-term substitutions is in the same logical form as the given argument. And conversely, every argument in the same logical form as a given argument is obtained from the given argument by a finite sequence of one-new-term substitutions—as long as the given argument involves only finitely many non-logical terms.

Extending this result to the case of arguments involving infinitely many non-logical terms is a mere technicality. Some people will want to take the above as a formal, "official", definition of the relation of "being-in-the-same-form-as". Compare with Corcoran (1989, "Argumentations and logic": 27ff).

Such concrete, material arguments should precede abstract, formal schemas, or schemata, such as the following.

Every N Rs itself.
Every A N Rs itself.

Every N x is such that xRx.
Every N x is such that if x is A, then xRx.

P
Q

Presenting argument schemas in the absence of their concrete instances alienates students from their native logical intuitions and gives them a distorted sense of logic. It has led to misconceptions such as that the primary subject matter of logic is logical forms or even schemata. It has even led to the view that logical reality excludes concrete arguments. It is also a mistake to call argument schemas by the expression *schematic arguments*: they are schemas and not arguments. Presenting argument schemas in the presence of their concrete instances is one of the practices I advocate under the rubric "contextualization". The same sentiment is in the 1981 *Preface to Quine* (1940):

> I used no schemata but referred only to their instances, the actual sentences, […]. I did not settle for open sentences, with free variables, but insisted on closed sentences, true and false. My reason was that these are what logic is for; schemata and even open sentences are technical aids along the way. (Quine 1940/1981, *Preface* iv).

Along with schemata and open sentences to be classified as "technical aids", Quine would have added logical forms if he had thought of it. To be

perfectly clear, I go a little further and say that I think teaching propositional logic first is a disservice to the students. Time has come to refute the myths that propositional logic is "primary", that it is presupposed by all other logics, and that it deserves some sort of exalted status. I do not teach propositional logic as a separate logic but as integral to basic logic. (Corcoran, 2001: "Second-order logic"). Moreover, I do not even mention "prothetic" or "quantified propositional logic"—which doesn't even make sense. See Section 6 below. As a first, introductory system of logic, I teach identity logic whose only logical constants are identity and inidentity. (Corcoran-Ziewacz, 1979: "Identity Logics"). Here are some examples of valid premise-conclusion arguments in identity logic.

$$+0 = 0$$
$$\underline{-0 = 0}$$
$$+0 = -0$$

$$+0 = 0$$
$$\underline{-0 = 0}$$
$$-0 = +0$$

$$+0 = 0$$
$$\underline{+0 \neq 1}$$
$$0 \neq 1$$

$$+0 = 0$$
$$\underline{+0 \neq 1}$$
$$1 \neq 0$$

$$\sqrt{0} = +0$$
$$+0 = -0$$
$$\underline{-0 = 0}$$
$$0 = \sqrt{0}$$

§2. History and philosophy

Here and elsewhere we shall not have the best insight into things until we see them growing from their beginnings.

—ARISTOTLE

Besides the new spirit there have been quiet developments in logic and its history and philosophy that could radically improve logic teaching.

Today more than ever before, we are alert to the human practices that gave rise to the living discipline we call logic: logic arises first as an attempt to understand proof or demonstration, alternatively—in a broader setting—to understand the axiomatic method and its presuppositions. This point of view is attested in the first paragraph of the book that marks the historical origin of logic: Aristotle's *Prior Analytics.* It is echoed, amplified, and updated in Cohen and Nagel's classic *Introduction to Logic* (Cohen-Nagel, 1934/1993). It is given an authoritatively mathematical restatement in the first paragraph of what is arguably the most successful and influential modern logic text: Alfred Tarski's *Introduction to Logic* (Tarski, 1941/1995). Alonzo Church's classic *Introduction to Mathematical Logic* (Church, 1956) makes a very closely related point on his first page. Aside from Galen and Sextus Empiricus, perhaps, this Aristotelian insight was largely ignored by logicians until Hilbert, Gödel, and others made it stand out. After World War II, Bourbaki's support of it was influential. (See Corcoran, 2009: "Aristotle's Demonstrative Logic").

As soon as the study of axiomatic method is undertaken, we slowly become aware that the same process of logical deduction used to obtain theorems from axioms is also used to obtain conclusions from arbitrary premises—whether known to be true or not. Evert Beth called this one of Aristotle's most important discoveries even though Aristotle never made the point explicitly, as far as I know.

Thus logic becomes a broader field: one whose aims include determining whether a given conclusion follows from given premises—or, what is the same thing, determining whether a given premise-conclusion argument is valid. Once this is undertaken, we see that the premises and conclusions need to be subjected to logical analysis—and that our methods of determining validity and invalidity need investigation, and so on.

The concern with determining whether a given conclusion follows from given premises—determining whether a given argument is valid—and the general methodology for approaching this concern is one of the perennial constants in logic which gets reconstructed and reaffirmed century after century starting with Aristotle. I have made this point in different ways in several papers. In Corcoran-Wood (1980), the very first paragraph reads as follows.

> It is one thing for a given proposition to follow or to not follow from a given set of propositions and it is quite another thing for it to be *shown* either that the given proposition follows or that it does not follow. Using a formal deduction to show that a conclusion *follows* and using a countermodel to show that a conclusion *does not follow* are both traditional practices recognized by Aristotle and used down through the history of logic. These practices presuppose, respectively, a criterion of validity and a criterion of invalidity each of which has been extended and refined by modern logicians: deductions are studied in formal syntax (proof theory) and countermodels are studied in formal semantics (model theory).

The method of countermodels, or counterinterpretations, which is for establishing invalidity, is a complicated and mathematically sophisticated form of Aristotle's method of counterarguments. The countermodel method has string-theoretic and set-theoretic prerequisites making it unsuited for elementary logic teaching. However, the counterargument method—used by Aristotle long before set theory or string theory were discovered—is well suited and, moreover, it lends itself to serving as an introduction to the method of countermodels.

What are the differences between the two methods? From a student's perspective, roughly speaking, in the method of counterarguments the meanings of the non-logical expressions are changed *by* changing their wordings—examples are given in Section 5 below—whereas in the method of countermodels the meanings of the non-logical expressions are changed *without* changing their wordings:

the wordings of the non-logical expressions are fixed but their meanings are changed—'reinterpreted' is a technical term often used. The method of countermodels requires separation of wordings from meanings, separating syntax from semantics, which is a prerequisite to the perplexing idea of reinterpretation of a language.

Admittedly, a historical perspective in logic teaching has been rare: Tarski, Church, and Quine notwithstanding. But, if my advice is followed, it will be increasingly emphasized in 21st century logic teaching.

Another related feature of 21st century logic teaching will be contextualizing. For example, it will not even be sufficient to see logic emerge in Aristotle's mind in response to his study of axiomatic method in Plato's Academy; it will be necessary to see Aristotle in his historical context: his predecessors and successors. To do that we could review the series: Thales, Pythagoras, Socrates, Plato, Aristotle, Euclid, Archimedes, Galen.

§3. Terminology

> The best notation is no notation; whenever possible, avoid complicated formalisms.
>
> —PAUL HALMOS

> Besides the new spirit, there have been quiet developments in logic and its history and philosophy that could radically improve logic teaching. One rather conspicuous example is that the process of refining logical terminology has been productive. Future logic students will no longer be burdened by obscure terminology and they will be able to read, think, talk, and write about logic in a more careful and more rewarding manner.

The goal of producing students who confidently and accurately think, speak, and write about logic is closely connected to the goal of producing students who can access the reality logic is about and who

can make autonomous judgments on logical issues. These goals are served by developing the ability to read logical writings—a skill that is not innate. Students must acquire it for themselves, but a teacher can help. One way a teacher can help students to acquire this skill is to read aloud to them important passages. As linguists know, but students often don't, the mind takes information more directly from spoken language than from the written. See my 2009 "Sentence, Proposition, Judgment, Statement, and Fact: Speaking about the Written English Used in Logic".

And do not fail to reread, sometimes two or more readings are needed to get the meaning to emerge. Also try to get the students to articulate what they experience. Encourage the students to see not only what the author is saying but also how the author said it: what choices, compromises, and trade-offs were made. And never fail to be interested in the students' interpretations and whether the students agree with the author's decisions.

At each stage of a logic course some passages will be more appropriate than others. Boole, De Morgan, Whitehead, Russell, Tarski, Cohen, Nagel, and Quine all produced models of lucid and rewarding passages. One of my favorites for beginning students is the section "Counterexamples and Proexamples" in the 1993 second edition of the classic Cohen-Nagel *Introduction to Logic,* page xxv. Other gems are scattered thoughout this paper.

A student asked why my list of logical-gem writers started with Boole. I could have gone back to Ockham, or even Augustine, but there are very few before that. It took a long time for our predecessors to learn how to write logic.

We no longer tolerate logicians who think they can escape criticism for confusing or misleading language by admitting to it in advance; an abuse of language isn't corrected by being identified in advance. Logicians don't earn our forgiveness by explicitly forgiving themselves. Using 'expression', 'term', 'concept', 'condition', and other words interchangeably is not conducive to learning. Likewise, 'sentence', 'proposition', 'judgment', 'statement', 'assertion', 'claim', and 'fact' all have their proper and separate ranges

of uses. (See Corcoran, 2009: "Sentence, Proposition, Judgment, Statement, and Fact").

However much a teacher may dislike a certain widely-used or traditional logic expression, that expression must be discussed in class in order for the students to be able to read the literature. Pretending that the expression doesn't exist—or that it is universally regarded by all competent logicians as a taboo expression—is not serving the student. Moreover, many such expressions are used differently by different logicians and the student needs to know this. An apt example is the word 'proposition', etymologically parallel to the Greek word *protasis* that Aristotle used for the things that could serve as premises and as conclusions of arguments. For example, if the word 'proposition' is not discussed with students, they will have trouble figuring out why Quine (1970) would spend so much time and energy trying to persuade people not to use the word. One useful paper—accessible with some help to beginning students—is my 2011 "Hare and Others on the Proposition".

The use-mention distinction, without which the Tarski truth-definition paper would have been inconceivable, is essential: '10' is a numeral, 10 is a number, and '10' denotes ten in Arabic base-ten notation—but '10' denotes two in binary or base-two notation. If use-mention cannot be done the first day, it should be done in the first week.

As important as the use-mention distinction is, even more important is the attitude that gave rise to it: the motivation to pursue logical reality and accuracy. People who appreciate the use-mention distinction, the logical analysis underlying it, and the terminology created to use it are also ready to seek further important distinctions and to seek higher levels of precision in logical writing. Another similar distinction is the sense-denotation dichotomy prominent in the writings of modern logicians such as Frege, Carnap, and Church—but already applied in the first sentence of Aristotle's *Categories*—which begins his *Organon* (Greek for "instrument"), a group of writings containing the first logic book. Another one is the type-token-occurrence distinction, a trichotomy that originated

in Peirce's writings and that is essential for clarity in discussing logic. See my paper on schemata (Corcoran, 2006: sect. 3, especially 228ff).

Any introduction to the literature of logic must warn students of obstacles such as inept and useless hijacking of entrenched normal language: logicians have been known to steal expressions they didn't need and would have been happier without. Mistakenly explaining 'is' as 'is identical with' is one example.

In normal English, 'Abe is Ben' means *roughly* "Abe is no-one but Ben": "Abe and Ben are one and the same person". Using Tarski's terminology, the sentence 'Abe is Ben' is true if and only if the name 'Abe' denotes the person Ben. To say that Abe and Ben are alike in relevant respects, 'Abe is identical to Ben' would be used. In fact, a person could say without raising eyebrows 'Abe is identical to Ben even though Abe thinks he is superior'. But in logic literature, 'Abe is identical to Ben' means that Abe is no-one but Ben, that Abe and Ben are one and the same person—uselessly employing 'identical to'. No logician could say 'Abe is identical to Ben even though Abe thinks he is superior': that would be practically a self-contradiction. Logicians are good at compartmentalizing: speaking English at home and "Loglish" at work. Aristotle set the precedent.

A closely related example is using 'equals' where 'is' belongs. Once this abuse of language is established it is awkward to make points such as that every side of an equilateral triangle equals both of the other two sides neither of which is the other. (See Corcoran-Ramnauth, 2013r.) It is by no means the case that using 'equals' for 'is' is ubiquitous in logic writing. Quine (1940) routinely used 'is' not 'equals', where identity is to the point.

No discussion of terminology would be complete without revealing nearly entrenched absurdities the students need to be warned of and explicitly excused from. The teacher must arrange class terminological and typographical conventions so that writing logic is not unnecessarily tedious. First, the student should be excused from the convention of italicizing variables instead of leaving them in roman—as if there were some sacred taboo that would be violated by writing 'every number x' in roman instead of 'every number *x*' in italic. This convention eats up a lot of time better spent on other things. Second, the student should be excused from compulsive use of quotes. Of course, it is important to make sure use-mention is observed, but there are other devices that make less clutter and use less time. Third, chose notation that is easy to read and easy to remember, e.g. for "and" use & instead of an inverted vee; for "the successor of" use ess as in y = sx instead of the accent as in y = x` (read ecks-accent, not ecks-prime). Under this rule comes minimizing the number of subscripts, superscripts, font changes, foreign alphabets, etc.

Also, simplify spelling: write 'premise' not 'premiss'. By the way, premises rhymes with cannabis, not canopies, and processes rhymes with auspices, not recipes: if you are ever tempted to be fancy—God forbid—check your dictionary. As Frango Nabrasa warns, mispluralizing English nouns as if they were Greek or Latin doesn't compensate for mispluralizing Greek or Latin nouns as if they were English. Your logic students trust you to be a model speaker of the language of instruction. Don't betray that trust.

§4. Variable-enhanced language

> The variable ranges over its values but is replaceable by its substituents. In arithmetic, the variable has numbers such as zero and one as its values but has numerals such as '0' and '1' as its substituents.
>
> —FRANGO NABRASA

> Closely related is increased use and study of variable-enhanced natural language as in "Every proposition x that implies some proposition y that is false also implies some proposition z that is true".

One variable-enhanced paraphrase of 'every person follows some person' is 'every person x follows some person y', but a more explicit paraphrase is 'every person x is such that x follows some person y'. The second occurrence of x is a *pronoun*

occurrence and the first marks the *antecedent referent* of the pronoun. The second occurrence refers back to the first. Every variable occurrence in a well-formed variable-enhanced English *sentence* is either a *pronoun* or an *antecedent*. But not every expression that resembles a sentence actually is a sentence, either having a truth-value or expressing a proposition having a truth-value. Consider 'x follows some person', where the pronoun lacks an antecedent referent as in the unenhanced 'he follows some person'.

Whenever there is a pronoun without an antecedent, the expression is not a *sentence* (expressing a *proposition*), although it could be a *predicate* (expressing a *condition*): 'x follows some person' expresses a condition satisfied by every person who follows some person. (See Tarski 1941/1995: sect.1, 5ff).

Every antecedent-occurrence of a variable is immediately after a common noun—the *range-indicator* for the variable. The common noun *person* is the range-indicator for the two occurrences of variable x in 'every person x is such that x follows some person y'. It is also the range-indicator for the occurrence of the variable y. But in many sentences there are different range-indicators for the occurrences of different variables as in 'every number x is denoted by some numeral y' or 'every number x is the length of some expression y'.

In many cases, roughly speaking, a range-indicator is to a variable as a common noun is to a pronoun. Church makes a similar point in Church (1956).

Whenever there is an antecedent without a pronoun, the expression can be made more explicit. For example, in the sentence 'every person x is such that x follows some person y', the last variable-occurrence is an antecedent having no pronoun referring back to it. The sentence can be made more explicit in multiple ways each having its own uses.

every person x is such that x follows y
    for some person y
every person x is such that, for some person y,
    x follows y

every person x is such that some person y
    is such that, x follows y
for every person x, some person y
    is such that x follows y
for every person x, for some person y,
    x follows y

It is even possible to get the pronoun be to its own antecedent.

every person x follows some person y

Anyway, there are several reasons for fine-tuning ones native ability to paraphrase into variable-enhanced language including, first, to understand better the logical form of the propositions expressed and, second, to prepare to translate into logically perfect languages, e.g., a symbolic formalized language. See "Logical form" in the *Cambridge Dictionary of Philosophy*, second and third editions.

It is my opinion that it is often easier to discern logical relations between propositions when they are expressed in variable-absent language than in fully explicit variable-enhanced language. However, it is often the case that logical relations are easier to discern using partly variable-enhanced language than either unenhanced or fully enhanced. But whatever opinion you may have, I hope you articulate it carefully and see what its consequences are and what might explain it.

On the subject of terminological transparency, whenever variables are introduced, constants should be introduced and the constant-variable distinction in logic and pure mathematics should be contrasted with the constant-variable distinction in science and applied mathematics. In logic and pure mathematics, constants and variables are symbols with contrasting sorts of meanings. In science and applied mathematics, constants and variables are not symbols but things, quantities with contrasting temporal behaviors.

My weight at this instant is a constant. My weight over this month is a variable. My age in years is a variable that is constant between birthdays. (See

Tarski, 1941/1995: 3). In the ordinary senses of 'variable', there is nothing variable about the variables in a given interpreted symbolic-language sentence or in a given variable-enhanced natural-language sentence. Logicians in the 21st century no longer say that such variables have variable meanings or that they denote variable things or that they denote ambiguously. Moreover, the fact that there may be contexts in which a variable is in some natural sense variable has nothing to do with why they are called variables.

Returning to the subject of validity, consider the following premise-conclusion arguments.

every person follows some person
every person follows some person
    who follows some person

every person follows some person
every person who follows some person
    follows some person

every person follows some person
every person follows some person
    who follows some person who follows some person

every person follows some person
every person who follows some person
    follows some person who follows some person

It is easy to see that each of these arguments is valid in the sense that its conclusion follows from its premises, i.e., that the conclusion simply brings out explicitly information already implicit in the premise—or at least does not add any information not in the premises—as explained in Corcoran (1998: "Information-theoretic logic"). Other logicians make similar points using other words. For example, Cohen and Nagel wrote the following.

The logical consequences of a proposition are not phenomena which follow it in time, but are rather parts of its meaning. While our apprehension of premises sometimes precedes that of their conclu-

sion, it is also true that we often first think of the conclusion and then find premises which imply it.

On the next page, they added: "That a proposition has definite logical consequences even if it is false follows also from the fact that these logical consequences or implications are part of its meaning". (Cohen-Nagel, 1934/1993: 9).

At this point some readers might ask, as one actually did:

Would you agree with the following? An argument is logically valid if and only if it takes a form that makes it impossible for the premises to be true and the conclusion nevertheless to be false.

I agree that an argument is valid iff every argument in the same form is valid. See Section 1 above. The ambiguous word 'form' is used in the sense of Corcoran (1989: "Argumentations and logic"), Quine (1970: *Philosophy of logic*), and others: every argument has exactly one form. I would also agree that an argument is valid iff it is logically impossible for the premises to be true and the conclusion false.

But I have some disagreements. First, a minor point of rhetoric: I would not qualify 'valid' with 'logically'; it would suggest that I recognize other sorts of validities. This in turn would raise the questions of what they are, what are the differences among them, and what they all have in common that justifies calling them validities. I prefer to set that to the side.

My important disagreement is with the naïve Platonistic suggestion that abstract logical forms are what *make* concrete arguments valid, that concrete arguments are valid *in virtue of* abstract form. I think this is destructive to clear thinking about logic; it has things backward in an alienating and oppressive way. A valid argument is made valid by the containment of its conclusion's information in its premise-set's information. To see whether a concrete argument is valid, students should be encouraged to understand its propositions and to see whether the conclusion's information can be extracted from that of the premises or whether the

conclusion's information goes beyond that of the premises.

What can we call the special property of abstract argument forms whose concrete instances are all valid? We cannot use 'validity' because that has been used for a property of concrete arguments. Calling a form valid would be a confusing category mistake: it would be ascribing to an abstract object a property applicable only to concrete objects. To use Peirce's example, it would be like saying that a color has a color, e.g. saying that green is green, i.e., that greenness has greenness, that green has greenness, that greenness is green.

I define an abstract argument form to be *omnivalid* if all of its concrete instances are valid; *nullovalid* if none are valid. Every argument form is omnivalid or nullovalid, since every two arguments in the same form are either both valid or both invalid.

I would add, paraphrasing Cohen-Nagel (1934/1993), that it is *not* the form that makes the argument valid; it is having valid instances that makes the form omnivalid: the form is omnivalid in virtue of its valid instances; the valid instance is not valid in virtue of its form. Cohen-Nagel (1934/1993) wrote the following on page 12.

> An argument is valid in virtue of the implication between premises and conclusion […] and not in virtue of […] the form which we have abstracted [sc. from it].

This is a good place to distinguish forms from schemata. (See Corcoran, 2006: "Schemata"). The instances of a form are all valid or all invalid. But there are schemata that have both valid instances and invalid instances. All one-premise arguments, whether valid or invalid, are instances of the following schema.

P
Q

Every one-premise argument having a negation for its premise, whether valid or invalid, is an instance of the following schema.

It is not the case that P
Q

Every one-premise argument having a negation as its conclusion, whether valid or invalid, is an instance of the following schema.

P
It is not the case that Q

I define a schema whose instances are all valid to be *panvalid*, whose instances are all invalid *paninvalid*, and those among whose instances are found both valid arguments and invalid arguments *neutrovalid*. (See Cohen-Nagel, 1934/1993, Editor's Introduction: xvii-xxxvii, especially xxxi ff).

Needless to say the class of concrete arguments has no members in common with either the class of forms or the class of schemata. Moreover, the latter two are also disjoint, i.e. the class of forms has no member in common with the class of schemata. At this point, I would warn against thinking of omnivalidity or panvalidity as a kind of validity—as 'validity' is used here and in my other writings.

In this work there is only one kind of validity and that is predicable only of concrete arguments. In the sense of 'valid' used here it would be an incoherency, a category mistake, to affirm or deny that something other than concrete arguments is valid. For the differences between logical forms and schemata see Cohen-Nagel (1934/1993, *Editor's Introduction*: xvii-xxxvii, especially xxxi ff). The distinction between an argument's unique form and its multiple schemata corresponds closely to Quine's distinction between a sentence's unique "entire structure" and its other "structure", for example, its truth-functional structure. (See Quine 1970, *Philosophy of Logic*: 48f; Also see Tarski-Givant, 1987: 43f).

Returning to the above four arguments that premise "every person follows some person", it would be interesting to discuss them and the infinitude of others constructed using the same transformations: i.e., taking a previously constructed

relative clause beginning 'who follows …' and inserting it after the noun 'person'.

But before going on we should express in variable-enhanced language the proposition expressed using the relative clause attached to the subject in the following.

every person who follows some person
    follows some person
every person x who follows some person y
    follows some person z
every person x who follows some person y
    is such that x follows some person z
every person x who is such that x follows some person y
    is then such that x follows some person z
every person x is such that if x follows some person y,
    then x follows some person z
every person x is such that if, for some person y,
    x follows y, then for some person z, x follows z

The above relative clauses are all restrictive, so called because, in typical cases, they restrict the extension of the noun-phrase they terminate: the extension of 'person who follows some person' is typically a proper subset of the extension of 'person'. Restrictive relative clauses are never set off by commas. But, as we learned in grammar class, there are attributive relative clauses that are always set off by commas and that are never parts of noun phrases.

every person leads some person
every person follows some person
every person, who follows some person,
    leads some person

every person leads some person
every person follows some person
every person, who leads some person,
    follows some person

every person, who leads some person,
    follows some person
every person follows some person

every person, who leads some person,
    follows some person
every person leads some person

every person, who leads some person,
    follows some person
every person leads some person and follows
    some person

The proposition—expressed using the comma—"every person, who leads some person, follows some person" contains exactly the same information as "every person leads some person and every person follows some person". In contrast, the proposition—expressed without the comma—"every person who leads some person follows some person" does not even imply "every person leads some person and follows some person". In fact, the proposition "every person who leads some person follows some person" is implied by "every person follows some person". But of course, the proposition "every person, who leads some person, follows some person" is not implied by "every person follows some person". For applications of these ideas to Peano and Gödel, see Sagüillo (1999: Sections 3.1 and 3.2).

There are many pitfalls in variable-enhanced English and many of those pitfalls are made even more dangerous when the inevitable sentence-abbreviating occurs. After all, variable-enhanced English isn't English and the "intuitions" that are reliable in English often need tweaking, or amending before being applied to variable-enhanced English. The following sentences express one and the same proposition: they all arise from variable-enhancement of the same plain-English sentence.

every person x follows some person y
every person y follows some person z
every person z follows some person x

There are various ways that a tyro can misunderstand these and conclude not only that they don't express the same proposition but that their propositions are logically independent as are the

following. The variables occupy places normally reserved for participles or something that can replace a participle.

every person walking follows some person jogging
every person jogging follows some person running
every person running follows some person walking

When a batch of variable-enhanced sentences all involve one and the same common noun, as these all involve only 'person', it is natural to leave the noun "understood". To read them, the noun must be restored "by the mind"—to use Tarski's terminology from his 1941 *Introduction*.

every x follows some y
every y follows some z
every z follows some x

A person's "English intuition" feels that the letters x, y, and z are nouns and the sentences are converted into spoken English somewhat as follows. [The word 'whigh' below names 'y' and rhymes with high, nigh, sigh, thigh, etc.]

every ecks follows some whigh
every whigh follows some zee
every zee follows some ecks

Misleading the student to think of variables as common nouns is even more likely when the variables are put into plural form as in 'all xs follow some ys' without an apostrophe or 'all x's follow some y's' with an apostrophe—pronounced "all eckses follow some whighs". There is another problem with pluralizing a variable using the apostrophe: that form is already used as a possessive as in 'if x is even, x's successor is odd'. To the best of my knowledge, no English noun pluralizes using apostrophe-ess.

Closely related to the mistake of treating variables as common nouns is the mistake of treating common nouns as proper names of sets. This can happen several ways: one is to write 'for every x ε person' instead of 'for every person x'; another even worse is to write 'for every x, if x ε person'. This is not grammatically correct variable-enhanced English. Moreover, it creates typographical clutter and it sets a scientistic tone inconsistent with autonomous judgment and independent thinking. We don't honor our heroes by mindlessly repeating their mistakes. In this case, Peano seems to be the originator of these mistakes (Quine, 1987).

The logic teacher will want to be alert for students falling into pitfalls. Every time a student falls into a pitfall, the teacher has an opportunity to instruct the class in the intricacies of variable-enhanced English and the logical analysis of English.

Logic teaching in the 21st century will look for opportunities to connect logic with other things the student has previously learned. For example, in language, composition, rhetoric, classics, history, and other classes, a student might learn Quintilian's four fundamental textual operations: addition, deletion, substitution, and transposition. These four words represent my interpretation of Quintilian's meanings not a translation of his words. Other English words are just as good: insertion, extraction, replacement, permutation. For operation, transformation would do as well. The Latin words Quintilian uses are *adiectio*, *detractio*, *mutatio*, and *ordinatio*.

The terminology is not fixed. Quintilian (96?/1920) observed that these four operations can be used to improve the rhetorical effectiveness of sentences (*Inst*. IX. iii. 27) and that they can be misused to undermine effectiveness and introduce errors (*Inst*. I. v. 8). The Latin words *Quadripartita Ratio* in the title of our journal are Quintilian's alluding to four transformations.

All four were used in this paper. The transitions to, from, and among the six variable-enhanced translations of 'every person follows some person' illustrate all four of Quintilian's transformation types: (1) addition of variables and (2) deletion, (3) substitution of 'every person x is such that' for 'for every person x', (4) transposition of 'for some person x'. These transitions call to mind the meaning-preserving transformations in Zellig Harris's "discourse analysis" that led via his student Noam Chomsky to

modern transformational grammars. (See Corcoran, 1972: "Harris on the Structures of Language").

The first two—under the names lengthening and shortening (or ellipsis)—are discussed and exemplified in several of my papers, (e.g. Corcoran, 2003: 266):

> Given two sentences expressing one and the same proposition, often one corresponds more closely to the logical form of the proposition than the other. Often one reveals more of the logical structure of the proposition or contains fewer logically irrelevant constituents. Some of the easiest examples of the grammatical–logical discrepancy are found in the so-called elliptical sentences that have been shortened for convenience or in the so-called expletive sentences that have been redundantly lengthened for emphasis or for some other rhetorical purpose.

Moreover, logic teaching in the 21st century will look for opportunities to make the student aware of the fact that logic can enrich the student's understanding of all previous learning. Awareness of logical issues can be like a sixth sense making other senses more vivid.

The issue of the attributive/restrictive distinctions is an apt example. Let us pause here to review some attributive/restrictive distinctions and the structural ambiguities requiring them. In this paper, when 'concrete' and 'abstract' are used with the common noun 'argument' they are used attributively, but when 'valid' and 'omnivalid' are used with the same common noun they are used restrictively. Thus, "Every concrete argument has its abstract form" is logically equivalent to "Every argument, which is concrete, has its form, which is abstract". However, "Every valid argument has its omnivalid form" is logically equivalent to "Every argument that is valid has its form that is omnivalid".

The adjective-noun phrase is structurally ambiguous. It has 'attributive' and 'restrictive' uses as explained in Corcoran (2009: "Ambiguity: Lexical and Structural").

In some cases, called attributive by grammarians, the implication is that the adjective applies to *everything* coming under the noun: "Every concrete argument has its abstract form" implies "Every argument is concrete" and "Every form is abstract". The point of attributive usage is often rhetorical, pedagogical, and expository: to remind the reader of an adjective previously applied to everything in the noun's extension—the extension of 'concrete argument' is the same as that of 'argument'.

In other cases, called restrictive by grammarians, the implication is *not* that the adjective applies to everything coming under the noun: "Every valid argument has its omnivalid form" does not imply "Every argument is valid" and it does not imply "Every form is omnivalid". In fact, to the contrary, as a matter of conversational implicature in the Grice sense, it suggests or "implicates" the opposite, i.e. "Not every argument is valid" and "Not every form is omnivalid". (See Grice, 1989: 24ff). The point of restrictive usage is often qualificational: to restrict the noun's extension—the extension of 'valid argument' is a proper subset of that of 'argument'. See Sagüillo (1999) and Corcoran (2009: "Ambiguity: Lexical and Structural").

As said above it is important to note, however, that although in *this* paper, whenever 'concrete' and 'abstract' are used with the common noun 'argument', they are used attributively, other works differ. That said, nevertheless, in this and every other work I can think of, whenever 'concrete' and 'abstract' are used with very general common nouns such as 'object', 'entity', 'individual', 'substance', etc., they are used restrictively. In fact, some writers seem to think that abstract objects and concrete objects are mutually exclusive and jointly exhaustive of reality.

Some older logic texts used the words *explicative* and *determinative* for occurrences of adjectives closely related to those called *attributive* and *restrictive* above. Roughly, whereas *attributive* and *restrictive* concern the structure of the proposition, *explicative* and *determinative* concern the structure of the reality being discussed—in the case of a true proposition, the structure of the fact. (See Watts, 1725/1790, *Logick*: Part II, ch. II, sect V).

The topic of structural ambiguity—also called amphiboly and amphibology—is a rich one whose

surface was hardly scratched above. In fact, there are many more things to teach and to learn about the structural ambiguity of the adjective-noun construction: every individual student is a student and, conversely, every student is an individual student. This example and those above bring to mind one of the most embarrassing chapters in the history of logic: the one titled "The law of inverse variation of intension and extension". (See Cohen-Nagel 1934/1993: 33).

Above I said that logic teaching in the 21st century will look for opportunities to connect logic with other things the student has previously learned. This section (§ 4) alone gives evidence of logic's relevance and applicability to grammar, rhetoric, and argumentation.

§5. Mathematical propositions, arguments, deductions, and counterarguments

> Since $x + 2 = 2 + x$ for any number x, it is true for some number x. Thus, as used here, any implies some and some does not exclude any.
>
> —WHITEHEAD (1911/1948: 8)

> Another welcome development is the culmination of the slow demise of logicism. No longer is the teacher blocked from using examples from arithmetic and algebra fearing that the students had been indoctrinated into thinking that every mathematical truth was a tautology and that every mathematical falsehood was a contradiction.

Our students already know some elementary mathematics. Logic teaching in the 21st century can follow Tarski's lead—in his *Introduction to Logic* (Tarski, 1941/1995)—by building on that knowledge, extending it, and using extensions of it to illustrate logical principles and methods. It is an insult to our students to teach as if a little elementary mathematics is beyond their abilities or worse that warm and

fuzzy examples will appeal to them. Our students already know the laws of commutativity and associativity of addition of integers in forms such as the following taken from elementary textbooks (Tarski 1941/1995: sect. 3).

C1: Commutativity: $x + y = y + x$
A1: Associativity: $(x + (y + z)) = ((x + y) + z)$

There are so many useful, important, and enriching things to say in a logic course about these laws of arithmetic it is hard to choose where to start. The first thing to do perhaps is to expand these highly-compressed elliptical sentences into variable-enhanced natural language. Tarski emphasizes that natural languages can express anything expressible in a formalized language and that there are many pedagogical advantages in translating a formula into natural language. In fact, in many passages he seemed to say that formalisms were abbreviations of colloquialisms.

C2: Where x and y are integers, x plus y is y plus x.

Since the initial sentence C1 has no singular/plural feature and since standard first-order sentences are generally translated using the singular grammatical "number", it is worth exploring a singular form.

C3: Where x is an integer, where y is an integer, x plus y is y plus x.

Do C2 and C3 express the same proposition as C1? Do C2 and C3 express the same proposition? Do C2 and C3 have the same consequences?

Is there any connection between the contrast of C2 with C3 and the contrast between the two-place quantifier $\forall xy$ and the one-place quantifier repeated $\forall x \forall y$ as in Tarski (1941/1995)?

The students will notice that the sentence C3 is very close to the sentence C4 below, where the second quantification comes at the end. They will also notice (1) that C4 is a little more natural and (2) that it exemplifies the fact that in variable-enhanced

language the quantifications often follow the variable-occurrences they bind.

C4: Where *x* is an integer, *x* plus *y* is *y* plus *x*, where *y* is an integer.

Asking the students why formalized language differs from natural language in quantification location alerts them to the phenomenon and at the same time extends the range of sentences they are comfortable symbolizing.

The propositions expressed by singular forms of the commutativity law clearly apply in the case of a single number that has two or more names: e.g., zero is named '+0' and '-0'. Thus, the students have no problem deducing '(+0 + -0) = (-0 + +0)' from C1, C3, or C4. In fact, to be clear, the following are both valid.

Where *x* is an integer, where *y* is an integer, *x* plus *y* is *y* plus *x*.
Where *x* is an integer, where *y* is the same integer, *x* plus *y* is y plus x.

Where *x* is an integer, where *y* is an integer, *x* plus *y* is *y* plus *x*.
Where *x* is an integer, where *y* is a different integer, *x* plus *y* is *y* plus *x*.

When conversing with beginners it is important to recognize and validate their insights. For example, some will notice that there is nothing to the premise of these two arguments besides the conclusions, i.e., the two conclusions together imply the premise: the following is valid.

Where *x* is an integer, where *y* is the same integer, *x* plus *y* is *y* plus *x*.
Where *x* is an integer, where *y* is a different integer, *x* plus *y* is *y* plus *x*.
Where *x* is an integer, where *y* is an integer, *x* plus *y* is *y* plus *x*.

The earlier it is in the course the more important it is for the teacher to explicitly draw the obvious conclusions, which are often eye-openers to the students: in this case the point to make is that the two premises of the last argument taken together are logically equivalent to the conclusion. The two premises just divide up the information in the conclusion; drawing the conclusion puts the information back together into one proposition, so to speak. (See Corcoran, 1995: "Information recovery problems").

However some students will guess, especially when helped with some Socratic questioning, that the commutativity proposition expressed by C2 beginning with the plural quantifier 'Where x and y are integers'—taken literally—does not imply:

(+0 + -0) = (-0 + +0).

In other words, they will guess that the following premise-conclusion argument, A1 below, is invalid—if the premise's sentence is read literally. Taken literally, the expression 'where *x* and *y* are integers' means the same as 'where *x* and *y* are different integers'. In cases when writers use it figuratively and do not want 'different' to be read in, they often add 'not necessarily distinct'. This brings a new set of interpretational problems: 'distinct' is not an adjective expressing a property of distinctness; it is elliptical for 'numerically distinct' expressing the relation of numerical distinctness and the whole added phrase is short for 'not necessarily numerically distinct from each other'. This interesting and important semantic issue plays no further role below.

ARGUMENT A1
Where *x* and *y* are different integers, *x* plus *y* is *y* plus *x*
+0 plus -0 is -0 plus +0

Continuing their train of thought, they will guess or maybe claim that the following is invalid.

ARGUMENT A2
Where *x* and *y* are different integers, *x* plus *y* is *y* plus *x*
Where *x* and *y* are the same integer, *x* plus *y* is *y* plus *x*

After all, some will say, each of the following arguments has a true premise but a false conclusion.

ARGUMENT B1
Where *x* and *y* are different integers, *x* exceeds *y*
or *y* exceeds *x*
+0 exceeds -0 or -0 exceeds +0

ARGUMENT B2
Where *x* and *y* are different integers, *x* exceeds *y*
or *y* exceeds *x*
Where *x* and *y* are the same integer, *x* exceeds *y*
or *y* exceeds *x*

In teaching, whenever an invalidity claim or guess is made, especially if it not obvious to everyone in the class, a counterargument should be given—preferably elicited from the class. Trying to find a counterargument for an argument that appears invalid can lead to a realization that appearances can be misleading and that the argument is actually valid. Notice that argument B1 is a counterargument to every other argument in its same form and to itself. The same holds for B2.

But B1 is not in the same form as A1: B1 has a relation "exceeds" but no operation, whereas A1 has an operation "plus" but no relation. They are however instances of one and the same neutrovalid schema: S1 below.

ARGUMENT SCHEMA S1
Where *x* and *y* are integers, R(*x, y*)
R(a, b)

For background experience, it is worth noting the validity of two other arguments.

Where *x* and *y* are different integers, *x* plus *y* is *y* plus *x*
if +0 isn't -0, then +0 plus -0 is -0 plus +0
Where *x* and *y* are different integers, *x* plus *y* is *y* plus *x*
Where *x* is an integer, *x* plus *y* is *y* plus *x*, where *y*
is an integer other than *x*

That being said a student might like to be reminded that the following is also valid.

Where *x* and *y* are different integers, *x* plus *y* is *y* plus *x*
where *x* is an integer, *x* plus *x* is *x* plus *x*

But the following is invalid, although +0 is -0.

Where *x* is an integer, *x* plus *x* is *x* plus *x*
+0 plus -0 is -0 plus +0

Of course if the premise is changed by adding 'and +0 is -0', the new argument would be valid. Judging the old argument as if it were the new would be the fallacy of premise-smuggling. (See Corcoran, 1989). The invalidity of the above is shown using the following counterargument.

Where *x* is an integer, *x* minus *x* is *x* minus *x*
+4 minus √4 is √4 minus +4

Deduction of the conclusion of Argument A1 from its premise, thereby establishing its validity, is a very easy exercise. Hint: take the tautology '+0 is -0 or +0 isn't -0' as the first line and use disjunctive reasoning. Once A1 is deduced, by adapting the same ideas, deducing Argument A2 will be easy. In a paper such as this, it is worth mentioning explicitly that 'deducing an argument' means "deducing its conclusion from its premise set"—as is natural and handy.

This discussion will give the instructor the opportunity to reiterate four important points. The first is that many excellent logic texts—including the influential 1934 Cohen-Nagel *Introduction* and even Tarski's 1941 masterpiece—treat plurals as singulars—and without a word of warning (Cohen-Nagel, 1934/1993: 42ff; Tarski, 1941/1995; 7ff).

The second is that *literal* reading of double universal quantifications expressed using pluralized range indicators—e. g., 'where *x* and *y* are integers'— is closely related to the "separated-variables" reading of double universal quantifications expressed using singular range indicators—'where *x*

is a number and *y* is a number'. The separated-variables reading takes the values assigned to the two variables to be two distinct numbers almost as if 'where *x* is a number and *y* is a number' were read as elliptical for 'where *x* is a number and *y* is a different number'. One reason for bringing this up is that some students are inclined to take it that way naturally—and thus to be out of touch with the class. I noticed this in my own teaching as have other logic teachers including Albert Visser (personal communication). Another reason is that Wittgenstein adopted a separated-variables approach in his 1922 *Tractatus Logico-Philosophicus.*

The third point the instructor can make is that finding inattention or even inaccuracy in a work is no evidence that alertness and exactness, perhaps even brilliant creativity is not to be found in it also. Don't throw the baby out with the bathwater. But, don't put the bathwater in the crib with the baby.

This reminds me of what Frango Nabrasa calls "Newton's Law of Fallacies": for every fallacy there's an equal and opposite fallacy. Trying too hard to avoid one lands you in the other. Falling backward is not a good way to avoid falling forward. "Political correctness" is not a good way to avoid ethnic, race, religion, philosophy, or gender insensitivity.

The fourth point, minor to the expert but eye-opening to the beginner, is that every integer has infinitely many names even if we don't count those made by adding any number of plus signs and those made by adding any even number of minus signs: $0 = - -0$, $0 = - - - -0$, etc.

This is a natural place to describe the pluralisms in logic that I advocate. The one I had in mind when I made the abstract and table of contents concerns awareness of the variety of classical logics actually used as underlying logics in traditional disciplines. In 1974 I discussed this pluralism and its role in historical research. See my 1974 "Future Research on Ancient Theories of Communication and Reasoning". The most important variety of classical logic by far is standard one-sorted, first-order logic. But many-sorted logic and higher-order logics are essential. See my 2001 "Second-order logic". Moreover, varieties of identity logics are useful for

understanding the logical experiences students have in their algebra courses. See the 1979 "Identity logics" and the 2015 "Teaching basic logics". We may call this *classical pluralism*: recognizing the variety of classical logics that can serve as underlying logics humans actually use in their intellectual lives.

Another form of pluralism I have advocated may be called *disciplinary pluralism*. This is the recognition that in the development of the many disciplines—arithmetic, geometry, set theory, etc.—there is no "one-size-fits-all" underlying logic; rather each classical discipline has its own classical underlying logic. For example, the logic of arithmetic differs from that of geometry in several ways. One metamathematically important way is that arithmetic has proper names for all of its objects but geometry has proper names for none of its objects—a point I learned from Tarski. See, for example, my 1973 "Gaps between logical theory and mathematical practice".

Applying this philosophy to teaching *excludes* presentation to undergraduates of "superlogics" such as the "functional calculus of first order" in Section 30 of Church (1956). These logics were constructed to achieve a maximum of generality but the result is alienating artificiality and ugly clutter. They have infinitely many classes of primitive symbols and each class is infinite. For example, for each number n, there are infinitely many n-placed predicate symbols. It would take years of study for a student to be able to see such "classical" logics as responding to any goal in classical logic implicit in the tradition founded by Aristotle. This kind of exclusion applies to many other "classical" logics including those in Tarski *1941/1995* and Quine (1970). I recommend that teachers avoid idiosyncratic, exotic, esoteric, artificial, unintuitive, or overly general forms of classical, Aristotelian, two-valued logics—call them what you want—to undergraduates. Try logics that help the student to discover logical reality and to get in touch with their own inner logician.

Another form of pluralism I have advocated may be called *analytical pluralism*. This is the

recognition that many a natural-language sentence used in a given discipline may be used to express different propositions and thus admit of a plurality of analyses: there is no "one-size-fits-all" logical analysis for a given sentence. Rather in each context one must do a new analysis—sometimes more than one in the same context. The question "what is the logical form of this sentence?" makes the usually-false presupposition that "this sentence" has only one logical form. We should ask "what are the logical forms of the propositions expressed by this sentence?".

For the record, I do not advocate teaching exotic, esoteric, non-classical, non-Aristotelian, deviant logics—call them what you want—to undergraduates who have not mastered articulations of their own inner logics. See Quine *1970* on deviant logics. This would be like teaching non-Euclidean geometries to students who had not yet developed their classical Euclidean intuitions. Maybe it would be more like teaching "languages" that were never used for communication and never will be. Again, perhaps it would be like feeding unhealthy commercial snacks to children who were still struggling to appreciate healthy home-cooked foods. There are other analogies that are even more negative.

Anyway, the pluralism that advocates teaching exotic non-classical logic to beginners is one I find counter-productive or worse. It alienates students from logical reality and prevents them from learning the logic they need in their lives. I call it *adventurist pluralism*.

That said, I hasten to add that I am far from condemning non-classical logics. That non-classical logics play fruitful roles in modern logical research is well established, as is explained, for example, in my 1973 "Gaps between logical theory and mathematical practice". Moreover, notice that I have not said classical logics are perfect models of human logical competence. On the contrary, I have said *that* they are not and *how* they are not—in the 1973 paper just mentioned and elsewhere—a point I will return to in my Conclusion below.

Moreover, let us not forget that our basic mathematical sciences presuppose classical underlying logics. For example, in number theory the great proofs including that of Fermat's Last Theorem use classical logic. Also, in mathematical logic, proofs of the great meta-theorems of Gödel, Tarski, Craig, and Henkin were not only classical in their reasoning but they were about classical logics.

Let us wrap up the discussion of commutativity and associativity by explaining how their independence is established using the method of counterarguments as described in various places including Corcoran (1989). The first step is to express them in full explicitly using a range-indicator: 'I' for 'integer'. To show that commutativity does not follow from associativity, consider the following.

$$\forall \mathsf{I}x \forall \mathsf{I}y \forall \mathsf{I}z(x + (y + z)) = ((x + y) + z)$$
$$\forall \mathsf{I}x \forall \mathsf{I}y(x + y) = (y + x)$$

The goal is to produce another argument in the same logical form with a premise known to be true and a conclusion known to be false.

For our universe of discourse, or range of values of our variables, we choose the strings of letters of the alphabet and take 'S', abbreviating 'String', as our range-indicator. Thus '$\forall \mathsf{S}x$' means "for every string x". For our two-place operation corresponding to addition we take concatenation: the result of concatenating the two-character string 'ab' to the three-character string 'cde' is the five-character string 'abcde'. Using the made-up word 'concat' for this operation, we can say that 'ab' concat 'cde' is 'abcde'. Using the arch '⌒' for "concat", we have the equation (identity):

$$\text{'ab'} \frown \text{'cde'} = \text{'abcde'}$$

Our counterargument is thus the following.

$$\forall \mathsf{S}x \forall \mathsf{S}y \forall \mathsf{S}z(x \frown (y \frown z)) = ((x \frown y) \frown z)$$
$$\forall \mathsf{S}x \forall \mathsf{S}y(x \frown y) = (y \frown x)$$

A little thought about strings reveals the truth of the premise. The falsity of the conclusion is seen by noting that it implies the following.

'ab' ∩ 'cde' = 'cde' ∩'ab'

But, 'abcde' isn't 'cdeab': the first begins with 'a', but the second begins with 'c'. Similar deliberations show that commutativity does not imply associativity.

The method of counterarguments was routinely and repeatedly used in practice almost instinctively before the theory used to describe it was developed. In fact, the method came before anyone mentioned logical forms of arguments. Indeed, the definition of being-in-the-same-form-as is of recent origin. See "Logical form" in Audi (2015). One of the theoretical principles presupposed by this method is that in order for an argument to be valid it is necessary and sufficient for every argument in the same form to be valid.

In teaching, the order of presentation should follow the historical order of discovery—at least this is a point Tarski stressed.

§6. Logical propositions, arguments, deductions, and counterarguments

> But many mathematicians seem to have so little feeling for logical purity and accuracy that they will use a word to stand for three or four different things, sooner than make the frightful decision to invent a new word.
>
> —FREGE 1893/1967, SECT. 60

> A further welcome development is the separation of laws of logic from so-called logical truths, i.e., tautologies. Now we can teach the logical independence of the laws of excluded middle and non-contradiction without fear that students had been indoctrinated into thinking that every logical law was a tautology and that every falsehood of logic was a contradiction. This separation permits the logic teacher to apply logic in the clarification of laws of logic.

Before treating the content of this topic it is necessary to reveal an embarrassing feature of the literature of logic. When a publication uses a familiar expression, the writer has certain responsibilities to the reader. Moreover when those responsibilities are not met, reviewers have the responsibility to point this out and to criticize the publication. The expressions of immediate relevance are *the law of non-contradiction*—some say *contradiction*, without the non—and *the law of excluded-middle*—some say *excluded-third*. Use of these without further explanation, especially in introductory contexts, presupposes that those expressions have fixed, generally agreed upon meanings and that the reader knows what those meanings are. Even if the publication explicitly says what these expressions are taken to denote, it is still inexcusably misleading not to warn the reader that these expressions have been used over centuries in many, perhaps a dozen or more, ways. Even worse, different senses are associated with different philosophies of logic.

Take the expression the *law of contradiction*. For centuries the ambiguous expression *Law of contradiction* (or *non-contradiction*) denoted (1) assertoric propositions such as

No proposition is both true and false,

(2) modalized versions with 'can be' for 'is'—and (3) very different modals such as

It is impossible that a property belonging
to an individual at
a time does not belong to the individual
at that same time.

This gives us three classes of uses, each containing two or more variants. But these three have been confused with others, three of which are mentioned here.

(4) No proposition is such that it and its negation are both true.

(5) No proposition is such that it and its contradictory are both true.

(6) No proposition is both true and not true.

However, Boole used the expression for an *equation* in class algebra, thus creating a seventh class of referents (Corcoran-Legault, 2013). This ambiguity persisted for decades—as Cohen and Nagel's popular and influential 1934 *Introduction* attests.

Using terminology from Tarski's *Introduction*, the first class has the variant:

No sentence is both true and false.

This law is unmistakably presupposed throughout Tarski's Chapters I and II, especially in Section 13 about truth-tables. Astoundingly, no such sentence occurs in Tarski's *Introduction*. Also conspicuously missing is an *explicit* statement that no sentential-function is satisfied and not satisfied by the same object. Absence of reference to any traditional law in Chapters I and II suggests the hypothesis that Tarski deliberately avoided mentioning it.

Another curious fact is that Tarski's Section 13 appropriated the expression *Law of contradiction* for a law which doesn't involve the words *true* and *false* or even symbolic renderings thereof—creating an eighth class of senses. Tarski abbreviated the law:

$$\sim[\, p \wedge (\sim p)]$$

Stated fully using Tarski's instructions (3,Section 13).

$$\text{for any sentence } p, \sim[\, p \wedge (\sim p)]$$

Another peculiarity is that Tarski avoids any clues about English translations of this perplexing sentence: its variables's values are exactly the same as their substituents—a peculiarity making the sentence difficult if not impossible to grasp. Having a variable's values being its substituents is a kind of use-mention conflation: a variable's substituents are *used* to *mention* its values. For example, in arithmetic, the individual variables have numbers as values and numerals as substituents: the number zero is a value of the variable having the numeral '0' as a substituent. Values are things in the universe of discourse of an interpreted language; substituents are names in that language.

Tarski's writing suggests, especially to beginners, that this strange and perplexing expression is what is normally called the *law of contradiction*.

Having dispensed some of our terminological responsibilities, let us turn to the main topic of this Section. The law of non-contradiction—"no proposition is both true and false"—and the law of excluded-middle—"every proposition is either true or false"—are both laws of logic but neither is a tautology, or logical truth in the broad sense. Every proposition in the same form as a tautology is a tautology and therefore a truth. But each of those two laws is in the same form as falsehoods: "no triangle is both equilateral and equiangular" is false and so is "every triangle is either equilateral or equiangular".

People who think that every law of logic is a tautology are apt to think that, since all tautologies are logically equivalent, all laws of logic are logically equivalent. But to see that noncontradiction doesn't imply excluded-middle it is sufficient to see that the following argument is invalid.

no proposition is both true and false
every proposition is either true or false

To see that this argument is invalid it is sufficient to see that it has a counterargument: an argument in the same form with a true premise and false conclusion.

no integer is both positive and negative
every integer is either positive or negative

To see that a universal proposition is false it is sufficient to see that it has a counterexample: in this case an object that satisfies the subject but dissatisfies the predicate. Zero is an *integer* that is not *either positive* or *negative*.

Thus noncontradiction does not imply excluded-middle. In other words, excluded-middle does not follow from noncontradiction; the argument having noncontradiction as its only premise and excluded-middle as its conclusion is invalid.

The same method shows that excluded-middle does not imply noncontradiction.

Incidentally, this example illustrates the importance of distinguishing counterargument from counterexample. But, this should not be taken to imply that no counterarguments are counterexamples. On the contrary, every counterargument for a given argument is a counterexample to the universal proposition that every argument in the same form as the given argument is valid.

Once methods and results have been presented, some succinct exercises are needed. Preferable exercises are that (1) maximize creative use of what has been learned and that (2) minimize writing. For these and other related reasons, alternative-constituent format questions are often appropriate. Here is one relevant example.

The law of (excluded-middle * noncontradiction) is logically equivalent to "every proposition that (is not * is) true (is not * is) false".

Alternative-constituent exercises can often be made more demanding as exemplified below.

The law of (excluded-middle * noncontradiction) is logically equivalent to "every proposition that (is not * is) (true *false) (is not * is) (false * true)".

The law of (excluded-middle * noncontradiction) is logically (equivalent to * independent of) "every proposition that (is not * is) (true *false) (is not * is) (false * true)".

Further discussion and application of the alternative-constituent format is found in my 2008 "Meanings of form" (Corcoran, 2009, and Corcoran-Main (2011).

Conclusion

As is evident by now to many readers, this essay does not intend to be definitive or comprehensive. It is more like a contribution to a dialogue. What did I leave out? Every reader will have an answer.

One glaring omission is the importance of memorization. My logical life has been enriched by reflecting on texts that I had memorized. Students have only the fuzziest idea of what the axiomatic method is unless they know of concrete examples. The first step in acquiring objectual knowledge of an axiom system is to memorize one. I require my students to memorize two axiom systems for arithmetic: the five Peano postulates and the three Gödel axioms used in his 1931 incompleteness paper. See the *Editor's Introduction* to Cohen-Nagel 1934/1993: *Introduction to Logic.* Once concrete examples are before the mind many questions come into focus and axiomatic method is promoted from being a topic of loose conversation to being an object for investigation.

I also recommend memorizing Euclid's axioms and postulates. These three examples of creative memorization are just the beginning.

Another important topic that has not been treated is something that has already been absorbed into logic teaching and that doesn't need to be recommended: teaching natural-deduction logic as opposed to axiomatic logic.

If I had more time, I would discuss the enormous mathematical, philosophical, and heuristic advantages of Jaśkowski -style sentential natural deduction. It is impossible to exaggerate the importance of Jaśkowski's insights—especially in my own thinking and research: I use them almost every day. See my three-part series Corcoran (1971) "Discourse Grammars and the Structure of Mathematical Reasoning".

Teaching a well-crafted, intuitive, and user-friendly Jaśkowski-style sentential natural deduction system can awaken a student's sense of logical reality and overcome the alienating effects of artificial approaches—truth-tables, trees, semantic tableaux, sequent calculi, Turing-machine implementable algorithms, etc.

There have been several small but important innovations in making natural deduction systems more natural. One is the recognition that indirect deduction is a special form of deduction not to be subsumed under negation intelim [sc. introduction-and-elimination] rules. Another is the recognition that deduction is a goal-directed activity and that goal-setting is an essential step. Both of

these points are developed in my 2009 "Aristotle's Demonstrative Logic", where special notational devices for indirect deduction and for goal-setting appear in print for the first time. It would be a mistake of the sort already criticized to think that currently available Jaśkowski-style systems cannot be made more realistic and thus more user-friendly.

Artificial approaches based on axiomatic logics, sequent logics, tree-logics, and the like are out of place in undergraduate logic. Such systems, of course, have their legitimate mathematical uses. For an interesting discussion, see Dummett (1973: 430ff). Moreover, knowledge of some of them is essential not only for certain advanced research but also for understanding the history of logic and the evolution of philosophy of logic. Nevertheless, as Michael Dummett emphasized in regard to axiomatic logics, their artificiality needs to be exposed so that a false view of logic is not conveyed as an officially-condoned viewpoint (Dummett, 1973: 432-434).

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Logic research in the 21st century is becoming more and more a communal activity as opposed to the solitary personal activity it was in the past. Before this century, with very rare exceptions, logical works were single-authored. In this century multiple-authored works are common and even single-authored works often have an acknowledgements section listing colleagues that contributed. Moreover, logic research in the 21st century is also becoming more and more an international activity. In fact, in this century we have multiple-authored works for which the authors are from different countries. Finally, logic research in the 21st century is becoming more and more interdisciplinary: logicians are listening more to the criticisms of logic made by their colleagues and logicians are responding to the logical needs of a broader community.

References

WORKS CITED

AUDI, R. (ed.) (2015). *Cambridge Dictionary of Philosophy* (third edition). Cambridge: Cambridge UP.

BOOLE, G. (1854/2003). *The Laws of Thought* (facsimile of 1854 edition, with an introduction by J. Corcoran). Buffalo: Prometheus Books.

CHURCH, A. (1956). *Introduction to Mathematical Logic.* Princeton: Princeton UP.

COHEN, M. AND E. NAGEL (1934/1993). *Introduction to Logic* (second edition, revised with new exercises, new indices, a new bibliography, and a 30-page introduction by J. Corcoran). Indianapolis: Hackett Publishing Company.

CORCORAN, J. (1972). "Harris on the Structures of Language". In *Transformationelle Analyse* (pp. 275-292) (ed. Senta Plötz). Frankfurt: Athenäum Verlag.

CORCORAN, J., W. FRANK AND M. MALONEY. (1974). String Theory. *Journal of Symbolic Logic* 39, 625-37. MR0398771 (53 #2622).

CORCORAN, J. AND J. SWINIARSKI (1978). Logical Structures of Ockham's Theory of Supposition. *Franciscan Studies,* 38, 161-183.

DUMMETT, M. (1973). *Frege: Philosophy of Language.* New York: Harper and Row.

FREGE, G. (1893/1967). *The Basic Laws of Arithmetic* (M. Furth, translator). Berkeley: UC Press.

GALEN. (200? /1964). *Institutio Logica.* (Trs. and ed. J. Kieffer). Baltimore: Johns Hopkins UP.

GRICE, P. (1989). *Studies in the Way of Words.* Cambridge MA: Harvard UP.

KNEALE, W. AND M. KNEALE. (1962). *Development of logic.* Oxford: Oxford UP.

LACHS, J. AND R. TALISSE (eds.) (2007). *American Philosophy: an Encyclopedia.* New York: Routledge.

QUINE, W. O. (1940). *Mathematical logic.* New York: Harper-Row.

QUINE, W. O. (1970). *Philosophy of logic.* Cambridge: Harvard UP.

QUINE, W. O. (1987). Peano as logician. *History and Philosophy of Logic*, 8, 15-24.

QUINTILIAN. (96?/1920). *Institutio Oratoria.* (Trans. H.E. Butler). Cambridge: Harvard UP.

RUSSELL, B. (1922). *Introduction to Wittgenstein's* Tractatus Logico-Philosophicus. London: Kegan Paul.

SAGÜILLO, J. M. (1999). "Domains of sciences, universe of discourse, and omega arguments". In *History and philosophy of logic, vol. 20* (pp. 267-280).

TARSKI, A. (1941/1995). *Introduction to Logic.* New York: Dover.

TARSKI, A. and S. GIVANT. (1987). *A Formalization of Set Theory without Variables.* Providence: American Mathematical Society.

WATTS, I. (1725/1790). *Logick.* London: Buckland *et al.*

WILLIAM OF OCKHAM. (1330?/1990). *Philosophical Writings* (Philotheus Boehner, translator), Indianapolis: Hackett.

WITTGENSTEIN, L. (1922). *Tractatus Logico-Philosophicus.* London: Kegan Paul.

LOGIC-TEACHING ARTICLES BY JOHN CORCORAN

(1971A). Discourse Grammars and the Structure of Mathematical Reasoning I: Mathematical Reasoning and Stratification of Language. *Journal of Structural Learning* 3(1), 55-74.

(1971B). Discourse Grammars and the Structure of Mathematical Reasoning II: The Nature of a Correct Theory of Proof and Its Value. *Journal of Structural Learning,* 3(2), 1-16.

(1971C). Discourse Grammars and the Structure of Mathematical Reasoning III: Two Theories of Proof. *Journal of Structural Learning* 3(3), 1-24.

(1972A). Conceptual Structure of Classical Logic. *Philosophy & Phenomenological Research,* 33, 25-47.

(1972B). Strange Arguments. *Notre Dame Journal of Formal Logic,* 13, 206-210. **MR0297513 (45 #6567).**

(1972C). Review of Quine 1970 "Philosophy of Logic". *Philosophy of Science,* 39, 97-99.

(1973A). "Gaps between logical theory and mathematical practice". In M. Bunge (ed.) *Methodological Unity of Science* (pp. 23-50). Dordrecht: Reidel Publishing Co.

(1973B). Meanings of Implication. *Diálogos,* 9, 59-76. MR95c:03019.

(1974). "Future Research on Ancient Theories of Communication and Reasoning". In J. Corcoran (ed.). *Ancient Logic and Its Modern Interpretations* (pp. 185-187). Dordrecht: Reidel Publishing Co. MR0485116 (58 #4971)**.**

(1979). Identity Logics. (Co-author: Steven Ziewacz). *Notre Dame Journal of Formal Logic,* 20, 777-84. MR0545427 (80H: 03017)

(1980). Boole's Criteria of Validity and Invalidity (Co-author: S. Wood). *Notre Dame Journal of Formal Logic,* 21, 609-39. MR0592521 (81J:03003)**.**

(1989A). Argumentations and Logic. *Argumentation,* 3, 17-43. MR91B:03006

(1989B). Significados de la Implicación. *Agora,* 5, 279-294. (Spanish translation by J. M. Sagüillo of "Meanings of Implication").

(1989C). The Inseparability of Logic and Ethics. *Free Inquiry.* Spring, 37-40.

(1994). Argumentaciones y lógica. *Agora,* 13(1), 27-55. (Translation by R. Fernández and J. Sagüillo of revised and expanded version of "Argumentations and Logic").

(1995, SEPTEMBER). Information recovery problems. *Theoria,* 10, 55-78. MR1361162 (96h: 03016)

(1995). Semantic Arithmetic: a Preface. *Agora,* 14(1), 149-156.

(1998). "Information-theoretic logic". In C. Martínez, U. Rivas and L. Villegas-Forero (eds.). *Truth in Perspective* (pp. 113-135). Aldershot, England: Ashgate Publishing Limited.

(1999). Critical thinking and pedagogical license. *Manuscrito,* 22, 109-116.

(2001). "Second-order Logic". In M. Zeleny and C. A. Anderson (eds.). "*Church Memorial Volume*", *Logic, Meaning, and Computation: Essays in Memory of Alonzo Church* (pp. 61-76). Dordrecht, Holland: Kluwer Publishing Company. MR2067966 (2005A:03021)

(2003). Aristotle's Prior Analytics and Boole's Laws of Thought. *History and Philosophy of Logic*, 24, 261-288.

(2006A). C. I. Lewis: History and Philosophy of Logic. *Transactions of the C. S. Peirce Society,* 42, 1-9.

(2006B). Schemata: the Concept of Schema in the History of Logic. *Bulletin of Symbolic Logic,* 12, 219-40.

(2008). Meanings of Form. *Manuscrito,* 31, 223-266.

(2009). Aristotle's Demonstrative Logic. *History and Philosophy of Logic,* 30, 1-20. Reviewed MR2488682 (2009m:03003).

(2009). "Sentence, Proposition, Judgment, Statement, and Fact: Speaking about the Written English Used in Logic". In W. A. Carnielli, M. E. Coniglio and I. M. Loffredo D' Ottaviano (eds.). *The Many*

Sides of Logic (pp. 71-103). (Series "Studies in Logic"). London: College Publications. MR2777861 (2012a:03010)

(2010). Los primeros días de todo curso de Lógica. *Ergo. Revista de Filosofía de la Universidad Veracruzana,* 25, 31-45. (Spanish translation by Patricia Diaz-Herrera of an unpublished paper "The first days of every logic course").

(2011). Hare and Others on the Proposition. *Principia,* 15(1), 51-76.

(2014). Existential import today: New metatheorems; historical, philosophical, and pedagogical misconceptions. *History and Philosophy of Logic,* 36, 39-61. (Co-author: Hassan Masoud). Published online 25 Sep 2014. *DOI:*10.1080/01445340.2014.952947

(2015a). Existential-import mathematics. *Bulletin of Symbolic Logic*, 21, 1-14. (Co-author: Hassan Masoud)

(2015b). "Investigating knowledge and opinion". In A. Buchsbaum and A. Koslow (eds.) *The Road to Universal Logic, vol. I.* (pp. 95-126). Basel: Springer. (Co-author Idris Samawi Hamid).

LOGIC-TEACHING ABSTRACTS BY JOHN CORCORAN

(1981). Ockham's Syllogistic Semantics. *Journal of Symbolic Logic,* 46, 197-198.

(1997). Teaching categoricity of arithmetic. *Bulletin of Symbolic Logic,* 3, 395.

(1999). The logical form of quantifier phrases: quantifier-sortal-variable. *Bulletin of Symbolic Logic,* 5, 418-419.

(2000). Self-defeating and self-fulfilling propositions. *Bulletin of Symbolic Logic,* 6, 111.

(2001). What are logical relations?. *Bulletin of Symbolic Logic,* 7, 105-106.

(2004). First-order Logical Form. *Bulletin of Symbolic Logic* 10, 445.

(2005). Meanings of word: type-occurrence-token. *Bulletin of Symbolic Logic*, 11, 117.

(2005). Counterexamples and Proexamples. *Bulletin of Symbolic Logic,* 11, 460.

(2005). Logically Equivalent False Universal Propositions with Different Counterexample Sets. *Bulletin of Symbolic Logic*, 11, 554-5.

(2006A). Complete enumerative inductions. *Bulletin of Symbolic Logic,* 12, 165-66.

(2006B). Meanings of Inference, Deduction, and Derivation. *Bulletin of Symbolic Logic,* 12, 353-354.

(2008A). Aristotle's many-sorted logic. *Bulletin of Symbolic Logic.* 14, 155–6.

(2008B). Iffication, Preiffication, Qualiffication, Reiffication, and Deiffication. *Bulletin of Symbolic Logic,* 14, 435-436.

(2009A). Agent and Premise Implication. *Bulletin of Symbolic Logic.* 15, 235. (Co-author: K. Barber).

(2009B). Alternative-Constituent Format. *Bulletin of Symbolic Logic.* 15, 133.

(2009C). Ambiguity: Lexical and Structural. *Bulletin of Symbolic Logic,* 15, 235-6.

(2009D). Consequence/consequent necessity. *Bulletin of Symbolic Logic,* 15, 461.

(2009E). Premise-conclusion Arguments. *Bulletin of Symbolic Logic,* 15, 461. (Co-author: J. Miller)

(2009F). Self-denying and self-affirming statements. *Bulletin of Symbolic Logic,* 15 (2009) 244.

(2009G). What is a proof?. *Bulletin of Symbolic Logic,* 15, 461-2. (Co-author: J. M. Sagüillo)

(2010). Hidden consequence and hidden independence. *Bulletin of Symbolic Logic,* 16, 443.

(2011). Numerically-indexed Alternative Constituent Format. *Bulletin of Symbolic Logic,* 17, 152-3. (Co-author: R. Main)

(2012A). Implications of implication. *Bulletin of Symbolic Logic.* 18 (2012) 147–8. (Co-author: J. Donhauser)

(2012B). Paracounterexamples: virtual counterexamples. *Bulletin of Symbolic Logic.* 18 (2012) 147.

(2012C). Predicates and predications. *Bulletin of Symbolic Logic,* 18, 148. (Co-author: C. McGrath)

(2012D). Quantifier-range-variable: the logical form of quantifications. *Bulletin of Symbolic Logic,* 18, 471.

(2012E). Refutation and proof. *Bulletin of Symbolic Logic,* 18, 473. (Co-author: H. Masoud)

(2012F). Truth-values and formal truth-value distributions of arguments. *Bulletin of Symbolic Logic,* 18, 148-9. (Co-author: S. Nambiar)

(2013A). Aristotle, Boole, and Tarski on contradiction. *Bulletin of Symbolic Logic,* 19, 515. (Co-author: J. Legault)

(2013B). "Deriving" Euclid's Interchange Rule from Leibniz's Law. *Bulletin of Symbolic Logic,* 19, 507-8.

(2013C). Equality and identity. *Bulletin of Symbolic Logic,* 19, 255-6. (Co-author: A. Ramnauth)

(2013D). Errors in Tarski's 1983 truth-definition paper. *Bulletin of Symbolic Logic,* 19, 514.

(2013E). Meta-language, object-language. *Bulletin of Symbolic Logic,* 19, 232-3. (Co-author: I. Samawi Hamid)

(2013F). Surprises in logic. *Bulletin of Symbolic Logic,* 19, 253. (Co-author: W. Frank)

(2013G). Surprising universal para-counterexample numbers. *Bulletin of Symbolic Logic,* 19, 255. (Co-author: S. Nambiar)

(2013H). Verifying and falsifying. *Bulletin of Symbolic Logic,* 19, 509-10. (Co-author: H. Masoud)

(2014A). Aristotelian logic and Euclidean geometry. *Bulletin of Symbolic Logic,* 20, 131. (Co-author: G. Boger)

(2014B). Conversely: extrapropositional and prosentential. *Bulletin of Symbolic Logic,* 20, 404. (Co-author: S. Nambiar)

(2014C). Cosmic Justice Hypotheses. *Bulletin of Symbolic Logic,* 20, 247-8. (Co-author: W. Frank)

(2014D). De Morgan on Euclid's fourth postulate. *Bulletin of Symbolic Logic,* 20, 250. (Co-author: S. Nambiar)

(2014E). Euclid's weak first axiom. *Bulletin of Symbolic Logic,* 20, 405. (Co-author: J. M. Sagüillo)

(2014F). Meanings of hypothesis. *Bulletin of Symbolic Logic,* 20, 248-9. (Co-authors: M. LaVine and J. Legault)

(2014G). Meanings of show. *Bulletin of Symbolic Logic,* 20, 403-4. (Co-author: J. Miller)

(2014H). Objectivity-subjectivity distinctions. *Bulletin of Symbolic Logic,* 20, 248. (Co-author: I. Samawi Hamid)

(2014I). Self-refuting propositions and self-sustaining propositions. *Bulletin of Symbolic Logic,* 20, 250. (Co-author: J. M. Sagüillo)

(2014J). Truth-preserving and consequence-preserving deduction rules. *Bulletin of Symbolic Logic,* 20, 130-1.

(2015A). Absence of argument-deduction-proof distinctions in Church 1956. *Bulletin of Symbolic Logic,* 21, 230-231.

(2015B). Aristotle's semiotic triangles and pyramids. *Bulletin of Symbolic Logic,* 21, 198.

(2015C). Deductive and inductive arguments. *Bulletin of Symbolic Logic,* 21, 231-232. (Co-author: J. M. Sagüillo)

(2015D). Expressing set-size equality. *Bulletin of Symbolic Logic,* 21, 239. (Co-author: G. Rising)

(2015E). Implication and deduction. *Bulletin of Symbolic Logic,* 21, 238-239. (Co-authors: H. Masoud and S. Nambiar)

(2015F). Plato's mathematical deductivism. *Bulletin of Symbolic Logic,* 21, 199. (Co-author: H. Masoud)

(2015G). Teaching course-of-values induction. *Bulletin of Symbolic Logic,* 21, 101.

(2015H). Teaching basic logics. *Bulletin of Symbolic Logic,* 21, 364.

(2015I). Teaching independence. *Bulletin of Symbolic Logic,* 21, 101-102. (Co-author: J. M. Sagüillo)